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PI-like Estimator-based Adaptive Extremum Seeking Control using Initial Excitation

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Abstract— **In this paper, a novel proportional integral (PI)-like estimator-based adaptive extremum seeking control (AdESC) algorithm is proposed for online optimization, where parameter convergence is achieved under a relaxed mathematical condition called initial excitation (IE). The proposed AdESC algorithm utilizes a new set of low-pass filter dynamics, which omits the requirement of switching mechanism in past literature for rank-checking while still ensuring parameter convergence. A detailed Lyapunov analysis is carried out using singular-perturbation like principle to establish closed-loop stability of the AdESC algorithm.**

I. Introduction

Extremum seeking control (ESC) has evolved as a promising feedback-based methodology, which tackles the problem of online optimization of an unknown objective/cost function with verifiable stability and robustness properties. The first detailed stability analysis of ESC were reported in [1], [2], using averaging and singular perturbation theory for ordinary differential equations. Furthermore, few other results on local stability of perturbation-based ESC scheme has been reported in [3], [4]. Extremum seeking has found applications in a wide variety of problems, including biochemical reactors [5], gas-turbine combustors [6], power electronics [7], etc.

Besides these applications, ESC algorithms have great potential in robot-based source seeking problems; especially for robots that operate in unknown/dynamic environments (i.e., homes, offices, elderly care centers). In such cases, ESC can be utilized to optimize some task variables of interest in real-time [8]–[10].

Adaptive ESC is an extension of ESC, where the optimizer simultaneously estimates the unknown cost and endeavors to locate the extremum-based on real-time sensing. Unbiased parameter estimation in steady state is a crucial demand in adaptive ESC (AdESC) since convergence to extremum directly relies on the estimation accuracy. Parameter convergence in AdESC using persistence of excitation (PE) condition is discussed in several works [5], [11], whereas a relaxed PE condition-based set-point problem is presented in [12]. Hybrid ES architectures that combine continuous-time and discrete-time dynamics, requiring a PE condition, is also presented in [13]. This PE condition is also vital many other control problems like model reference adaptive control [14], [15], adaptive optimal control [16], etc. However, the condition is stringent in nature due to its lack of practical feasibility. The condition relies on the future behavior of the signal, making it difficult to verify online. Persistent perturbations are often injected to the controller for sufficient exploration to satisfy the PE condition. While this exploration technique may improve estimation efficiency, it hampers the actual control objective in many settings [14]. Recently, a new technique called concurrent learning (CL) is reported in [17], [18], which claims that the classical PE condition can be relaxed using "sufficiently rich" recorded data. Along the similar lines, some recent works [19]–[22] have proposed two-tier filter-based adaptive controllers to ensure parameter convergence using an online-verifiable condition of initial excitation (IE). The IE condition is shown to be effectively milder than the PE condition since it demands sufficient information content only in an initial time window (transient period), unlike PE. Second-layer filter architecture in IE-based framework has two different versions open-loop [19], [23] and closed-loop [20], [22], respectively. While CL as well as closed-loop version of IE uses a computationally burdensome rank condition for switched adaptation, open-loop version of IE framework lacks internal stability and robustness to external disturbance.

In this paper, a novel proportional integral (PI)-like estimator-based AdESC algorithm is proposed for online optimization, where parameter convergence is achieved under the relaxed excitation condition called initial excitation (IE). Unlike previous IE-based results, a novel weighting function-based filter is introduced here. The newly introduced filter dynamics does not have instability issue as well as online rank-checking requirement (IE verification), which were the major concerns in open-loop and closed-loop filter architectures of IE-based designs [19], [22], respectively. A rigorous stability analysis with a novel choice of Lyapunov function candidate is performed using singular perturbation theory, which ensures global exponential convergence of the concatenated error dynamics to the equilibrium point. Simulation results further validate the efficacy of the proposed filter based novel AdESC algorithm.

II. Problem Formulation

A. Model Description

Consider an autonomous agent with single-integrator dynamics $\forall t \geq t_0$

$$\dot{x}(t) = u(t), \tag{1}$$

$$y(t) = J(x(t)) \tag{2}$$

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where $x(t) \in \mathbb{R}^n$ denotes position, $u(t) \in \mathbb{R}^n$ denotes velocity control input, $y(t) \in \mathbb{R}$ denotes output/sensory performance described by a performance index/cost function $J : \mathbb{R}^n \to \mathbb{R}$, which is an unknown continuously differentiable function defined on an open set containing a closed set $M \subset \mathbb{R}^n$ of interest. It is assumed that the agent can only access the performance index $J(x)$ at its own coordinate.

The following assumptions on the unknown performance index $J(x)$ are considered to facilitate the subsequent development.

Assumption 1. *The performance index $J(x)$ can be linearly parametrized i.e., $\exists$ an unknown constant parameter vector $\theta \in \mathbb{R}^p$ and a known continuously differentiable regressor function $\phi(x(t)) \in \mathbb{R}^p$ (plausibly non-linear with the property that $(x(t) \in \mathscr{L}_\infty \implies \phi(x(t)) \in \mathscr{L}_\infty)$ such that*

$$J(x(t)) \triangleq \theta^{\mathrm{T}} \phi(x(t)) \tag{3}$$

□

Assumption 2. *Performance index $J(x)$ is strongly convex with modulus $\mu > 0$ and continuously differentiable on an open set $D \supset \bar{M} \supset M$, and the set M and $\bar{M}$ are compact, convex, and nonempty sets to be defined subsequently. Moreover, $\nabla J(x)$ is Lipschitz on D with Lipschitz constant $L > 0$.* □

B. Objectives

The objective of this work is to develop a filter-based AdESC algorithm i.e., a pair of control input $u(t)$, parameter estimator $\hat{\theta}(t)$ are to be designed such that the following arguments hold:

1) $\|x(t) - x^\star\| \to 0$ *as* $t \to \infty, \ \forall t \geq t_0$

with

$$x^\star = \operatorname{argmin} J(x), \quad \text{subject to } x(t_0) \in M \tag{4}$$

2) $\|\hat{\theta}(t) - \theta\| \to 0$ *as* $t \to \infty, \ \forall t \geq t_0$ (5)

where $\hat{\theta}(t) \in \mathbb{R}^p$ denote the estimate of unknown constant vector θ.

In general, classical AdESC [5], [12], [24] and data-enabled (ESC) [25], [26] algorithms require the restrictive PE condition and an online-verifiable computationally-expensive rank condition on the stored data for parameter convergence, respectively. However, to obviate such issues, this work relies on online measurements while capturing past information via novel integral action.

III. Filter-based PI-like AdESC algorithm design

A. Integral Filter Architecture

To design a novel filter-based PI-like parameter estimation algorithm, the following low-pass filter equations are introduced, $\forall t \geq t_0$.

$$\dot{Y}(x(t)) = \alpha(t)\phi(x(t))\phi^{\mathrm{T}}(x(t)), \ Y(x(t_0)) = 0 \tag{6}$$

$$\dot{Z}(x(t)) = \alpha(t)\phi(x(t))J^{\mathrm{T}}(x(t)), \ Z(x(t_0)) = 0 \tag{7}$$

where $Y(x(t)) \in \mathbb{R}^{p \times p}$ denotes the filtered regressor, $Z(x(t)) \in \mathbb{R}^p$ denotes the filtered output, and $\alpha(t) \in \mathbb{R}$ is strategically introduced as a weighting function, which has the following properties:

$$\alpha(t) > 0, \ \forall t \in [t_0, \infty) \tag{8}$$

$$\alpha(t) < \bar{\alpha} < \infty, \ \forall t \in [t_0, \infty), \ \alpha(t) \in \mathscr{L}_\infty \tag{9}$$

$$\alpha(t) \in \mathscr{L}_1 \tag{10}$$

where $\bar{\alpha} \in \mathbb{R}_{>0}$ is the upper-bound of $\alpha(t)$.

Remark 1. *Novelty of the filter dynamics (6)-(7) relies on the weighting function $\alpha(t) \in \mathbb{R}$, which satisfies the above properties (8)-(10). This facilitates the boundedness property of $Y(x(t))$ and $Z(x(t))$ as revealed later in the stability analysis section.* □

Analytically solving the (6), (7), along with (3), it can be deduced that

$$Z(x(t)) \triangleq Y(x(t))\theta, \ \forall t \geq t_0 \tag{11}$$

The following properties of the filtered regressor $Y(x(t))$ are crucial for the subsequent development.

Property 1. *$Y(x(t))$ is a positive semi-definite function of time i.e., $Y(x(t)) \geq 0, \ \forall t \geq t_0$.* □

Proof: From (6), the square matrix $Y(x(t))$ can be represented as

$$Y(x(t)) = \int_{t_0}^{T} \underbrace{\alpha(r)}_{>0} \underbrace{\phi(x(r))\phi^{\mathrm{T}}(x(r))}_{\geq 0} \mathrm{d}r \tag{12}$$

Utilizing (8), in (12), it can be deduced that $Y(x(t)) \geq 0$, $\forall t \geq t_0$. ■

Property 2. *$Y(x(t))$ is a non-decreasing function of time in the sense of matrix inequality i.e., $Y(x(t_2)) \geq Y(x(t_1))$, $\forall t_2 \geq t_1 \geq t_0$*

Proof: From (6), the square matrix $Y(x(t))$ can also be expressed as

$$Y(x(t_2)) = Y(x(t_1)) + \underbrace{\int_{t_1}^{t_2} \alpha(r)\phi(x(r))\phi^{\mathrm{T}}(x(r))\mathrm{d}r}_{\geq 0} \tag{13}$$

From (13), utilizing the Property 1, it can be concluded that $Y(x(t_2)) \geq Y(x(t_1)), \ \forall t_2 \geq t_1 \geq t_0$. ■

To capture past information via integral action, The following Assumption on regressor function $\phi(x(t))$ is considered, which needs to be satisfied by the trajectory $x(t)$.

Assumption 3. *The regressor function $\phi(x(t)) \in \mathbb{R}^p$ is uniformly initially exciting (u-IE) w.r.t the dynamics (1), i.e., $\exists\, \Upsilon, T > 0$, such that, $\forall (t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, all corresponding solutions satisfy*

$$\int_{t_0}^{t_0+T} \phi\big(x(r,t_0,x_0)\big)\phi^{\mathrm{T}}\big(x(r,t_0,x_0)\big)\mathrm{d}r \geq \Upsilon I_p \tag{14}$$

□

Remark 2. *The IE condition (defined in [19]–[22]) requires the excitation/richness only in the initial finite time window, which is significantly less restrictive than the PE condition, where the excitation is needed for the entire time span [27],* *[28]. It is argued in [20] that the notion of persistence is entirely abolished owing to the richness requirement merely in the initial time-window, i.e., the excitation need not sustain as the window is moved forward in time. It is observed in [19]–[22] that a frequency-rich transient perturbation facilitates the IE condition without drastically affecting tracking performance, unlike PE, which requires frequency-rich persistent perturbation, hampering tracking performance. Since the IE-based adaptive controllers establish exponential convergence (a stronger notion of convergence), they in turn provide superior transient tracking performance (heuristically) in contrast to classical adaptive controllers as verified through extensive simulation studies in [19]–[22]. This work exploits the benefit of IE condition for AdESC with the use of a novel filtering mechanism* (6)-(7). □

Lemma 1. *Provided Assumption 3 holds, $Y(x(t)) \in \mathbb{R}^{p\times p}$ is positive definite, $\forall t \geq t_0 + T$.*

Proof: Utilizing the property (9) of weighting function $\alpha(t)$, and (12), the filtered-regressor can be lower-bound as

$$Y(x(t_0+T)) \geq \underline{\alpha} \int_{t_0}^{t_0+T} \phi(x(r))\phi^{\mathrm{T}}(x(r))\mathrm{d}r, \tag{15}$$

where $\underline{\alpha} > 0$ is the lower-bound of $\alpha(t)$, $\forall t \in [t_0,\ t_0+T]$.

From (15), utilizing Assumption 3, it can be deduced that

$$Y(x(t)) \geq \underline{\alpha}\Upsilon I_p,\ \forall t \geq t_0+T. \tag{16}$$

From (16), it can be concluded that $Y(x(t)) \in \mathbb{R}^{p\times p}$ is positive definite, $\forall t \geq t_0 + T$. ■

In order to satisfy the objectives (4)-(5), consider an AdESC algorithm characterized by the following parameter estimator and controller duo.

B. Proposed PI-like Parameter Estimator

Consider the novel PI-like parameter estimation law

$$\begin{aligned}\dot{\hat{\theta}}(t) =& F_\theta(\hat{\theta},x)\\ =& -\gamma\Gamma_\theta\phi(x(t))\underbrace{\big(\hat{J}(x(t))-J(x(t))\big)^{\mathrm{T}}}_{T_{pf}}\\ & -\gamma_1\Gamma_\theta\underbrace{\big(\hat{Z}(x(t))-Z(x(t))\big)}_{T_{if}},\ \forall t\geq t_0\end{aligned} \tag{17}$$

with

$$\hat{J}(x(t)) \triangleq \hat{\theta}^{\mathrm{T}}(t)\phi(x(t)),\ \hat{Z}(x(t)) \triangleq Y(x(t))\hat{\theta}(t)$$

where $\Gamma_\theta \in \mathbb{R}^{p\times p}$ is a positive definite learning gain matrix, γ and γ_1 are positive tuning parameters. The proposed PI-like parameter estimator dynamics in (17) consists of the term T_{pf}, which is proportional-type prediction-error; and T_{if} is the newly introduced integral-type prediction-error.

C. Proposed Controller

Consider the following controller

$$u(t) = \varepsilon\Big[-x+P_M\big(x-\eta\hat{\theta}^{\mathrm{T}}\nabla\phi\big(x(t)\big)\big)\Big]+u_{\mathrm{ex}}(t),\ \forall t\geq t_0 \tag{18}$$

where $\varepsilon \in \mathbb{R}_{>0}$ is a controller tuning parameter; projection operator is defined as $P_M(x) = \arg\min_{u\in M}\|x-u\|_2$, and $u_{\mathrm{ex}}(t)$ is an exponentially decaying probing signal having the following properties that, $u_{ex}(t) \in \mathscr{L}_\infty$ and $u_{\mathrm{ex}}(t) \to 0$ *as* $t \to \infty$. $\eta \in \mathbb{R}_{>0}$ is a design parameter. The purpose of embedding such probing signal $u_{\mathrm{ex}}(t)$ is to provide sufficient excitation to achieve parameter convergence [5], [12], [29], which helps the algorithm to get unknown minimum $x^\star$, θ of the objective (4) and (5), respectively. However, the notion of decaying exploration is unique in the proposed work as compared to past literature on ESC, where persistent exploration is considered to satisfy the PE condition.

Utilizing (18) in (1), the agent dynamics in closed-loop form can be expressed as

$$\begin{aligned}\dot{x}(t) &= \varepsilon F_x(x,\hat{\theta})+u_{\mathrm{ex}}(t)\\ &= \varepsilon\Big[-x+P_M\big(x-\eta\hat{\theta}^{\mathrm{T}}\nabla\phi\big(x(t)\big)\big)\Big]+u_{\mathrm{ex}}(t),\ \forall t\geq t_0\end{aligned} \tag{19}$$

D. Closed-loop Stability Analysis

Due to the coupled nature of the estimator and the controller, the analysis demands to use singular perturbation theory [30]-like method, where estimator dynamics (17) should have a faster time-scale than the dynamics (19)[1]. The two-time scale analysis is presented below.

By introducing $\tilde{\theta}(t) := \hat{\theta}(t)-\theta$, $\tilde{x}(t) := x(t)-x^\star$, where $(\theta,x^\star)$ is the equilibrium point of the pair $\big((17),(19)\big)$ and expressing the closed-loop equations in $\tilde{\theta}$, $\tilde{x}$ coordinates and in the new time scale $\tau = \varepsilon t$, the model of the closed-loop system is obtained in the standard singular perturbation form as $\forall t \geq t_0$

$$\varepsilon\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\tau} = F_\theta(\tilde{\theta}+\theta,\ \tilde{x}+x^\star) \tag{20}$$

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tau} = F_x(\tilde{x}+x^\star,\ \tilde{\theta}+\theta)+\frac{1}{\varepsilon}u_{\mathrm{ex}}(\frac{\tau}{\varepsilon}) \tag{21}$$

where (20) and (21) denote as boundary layer/fast dynamics and slow dynamics, respectively.

The following Theorem provides stability guarantees for the parameter estimation algorithm with fast dynamics (20).

Theorem 2. *The boundary layer (fast) dynamics*

$$\dot{\tilde{\theta}} = F_\theta(\tilde{\theta}(t)+\theta,\ \tilde{x}(t)+x^\star) \tag{22}$$

satisfies the following properties.

1) *Stability/Convergence: The origin* ($\tilde{\theta} = 0$) *of the boundary layer (fast) dynamics is Lyapunov stable and all the auxiliary signals remain bounded for all time.*

[1]Throughout the paper, it is assumed that all the functions are sufficiently smooth so that appropriate singular perturbation theory can be used.

2) *Prediction-error convergence:*

$$T_{pf}(t) \triangleq \left(\hat{J}(x(t)) - J(x(t))\right) \to 0 \;\; as\;\; t \to \infty$$
$$T_{if}(t) \triangleq \left(\hat{Z}(x(t)) - Z(x(t))\right) \to 0 \;\; as\;\; t \to \infty.$$

3) *Parameter convergence under u-IE: Provided the result of Lemma 1, then the origin of the boundary layer dynamics $\tilde{\theta}(t)$ is uniformly globally exponential stable (UGES) in a delayed sense i.e.,*

$$\|\tilde{\theta}(t)\| \le \delta_1 \|\tilde{\theta}(t_0+T)\| e^{-\delta_2(t-t_0-T)}, \;\forall t \ge t_0+T \quad (23)$$

where $\delta_1, \delta_2 \in \mathbb{R}_{>0}$.

Proof: Consider the following Lyapunov candidate for the boundary layer (fast) dynamics

$$V_1(\tilde{\theta}(t)) = \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma_{\theta}^{-1}\tilde{\theta}. \quad (24)$$

Taking the time derivative of (24) along (22), yields

$$\dot{V}_1 = -\tilde{\theta}^{\mathrm{T}}\left(\gamma\phi\phi^{\mathrm{T}} + \gamma_1 Y\right)\tilde{\theta} \le 0. \quad (25)$$

Using the Property 1 in (25), the above inequality implies Lyapunov stability of the error dynamics $\tilde{\theta}(t)$. Utilizing the result of Lemma 1 which is based on Assumption 3, $\dot{V}_1$ in (25) can be further upper bounded as

$$\dot{V}_1 \le -\underline{\alpha}\Upsilon\gamma_1\|\tilde{\theta}(t)\|^2 \le 0, \;\forall t \ge t_0+T \quad (26)$$

and after using (24), the inequality (26) can be written as,

$$\dot{V}_1 \le -2\underline{\alpha}\Upsilon\gamma_1 V_1, \;\forall t \ge t_0+T. \quad (27)$$

Hence using the comparison Lemma of [30], the differential inequality in (27) leads to the subsequent exponentially convergent bound on V_1:

$$V_1(t) \le V_1(t_0+T)e^{-2\underline{\alpha}\Upsilon\gamma_1(t-t_0-T)}, \;\forall t \ge t_0+T. \quad (28)$$

From (24), one has $\|\tilde{\theta}(t)\| = \sqrt{2V_1(t)}$, which implies that $\|\tilde{\theta}(t)\|$ is exponentially convergent to zero for $t \ge t_0+T$, i.e., (23) holds true. Since $V_1(t)$ in (24) is radially unbounded, the result holds globally. ■

Theorem 3. *Provided that Assumption 2 holds, neglecting the effect of exponential decaying perturbation signal i.e., $u_{ex}(t) = 0$, the origin ($\tilde{x} \equiv 0$) of the slow dynamics $\forall t \ge t_0$*

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tau} = F_x(\tilde{x}+x^\star,\; \theta) \quad (29)$$

is uniformly globally exponentially stable.

Proof: This follows the proof argument of Theorems 3-4 in [31], [32]. ■

Remark 3. *Based on [31], [32], Theorem 3 implies that, for each $x(t_0) \in M$, there exist a unique solution $x(t) \in M$, $\forall t \ge t_0$. However, (19) with $u_{ex}(t) = 0$, also implies same argument i.e., $x(t) \in M$, $\forall t \ge t_0$, followed from [31], [32] along with [25]. In the same line, due to the presence of exploratory signal $u_{ex}(t)$ in (19), it can be insured that for each $x(t_0) \in M$, there exist a unique solution $x(t) \in \bar{M}$, $\forall t \ge t_0$ where $M \subseteq \bar{M} \subset D$, and the increase in diameter of $\bar{M}$ as compared to M is depend on the size of exploratory signal $u_{ex}(t)$.* □

Above individual results on fast and slow dynamics dictate that Tikhonov's Theorem (Theorem 11.3 [30])-like arguments can be invoked for proving asymptotic stability of the equilibrium point of the coupled dynamics, comprising parameter estimator dynamics (17) and agent dynamics (19). In accordance with the above, stability property of the coupled system is stated in the subsequent theorem additionally revealing the condition for exponential convergence, while including the effect of exploratory signal on the closed-loop dynamics.

Theorem 4. *IF Assumptions 1-3 hold, the filter-based PI-like AdESC algorithm exhibit the following properties.*

1) *For the parameter estimator dynamics (17), $\hat{\theta}(t)$ converges to the unknown constant θ exponentially (objective (5) is achieved);*
2) *The dynamics (19) with $x(t_0) \in M$ converge to the unique solution $x^\star$ exponentially (optimization objective (4) is achieved);*

provided that the following gain conditions are satisfied:

$$0 < \varepsilon < \frac{\mu}{\eta^2(L^2+1)}, \;\; \underline{\alpha}\Upsilon\gamma_1 > \frac{1-d}{d}C^2.$$

Proof: Consider the following composite Lyapunov candidate

$$V = d\underbrace{\left(\frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma_{\theta}^{-1}\tilde{\theta}\right)}_{V_1} + (1-d)\underbrace{\left(J(x) - J(x^\star) + \frac{1}{2}\|x-x^\star\|^2\right)}_{V_2} \quad (30)$$

where $d \in (0,1)$.

Taking the time derivative of (30) along with the (17) and (19), yields

$$\begin{aligned}\dot{V} \le &-d\underline{\alpha}\Upsilon\gamma_1\|\tilde{\theta}\|^2 - (1-d)\left(\nabla J(x) + (x-x^\star)\right)^{\mathrm{T}}\varepsilon\Big((x-\bar{x})\\ &- P_M\left(x - \eta\hat{\theta}^{\mathrm{T}}\nabla\phi\right) + P_M\left(x-\eta\theta^{\mathrm{T}}\nabla\phi\right)\Big)\\ &+ (1-d)\left(\nabla J(x) + (x-x^\star)\right)^{\mathrm{T}}u_{\mathrm{ex}}(t), \;\forall t \ge t_0+T \end{aligned} \quad (31)$$

where $\bar{x} := P_M\left(x-\eta\theta^{\mathrm{T}}\nabla\phi\right)$. Within a compact set ($\bar{M}$) there exist $C > 0$ such that $\|\nabla\phi\| \le C$. Further using the proof argument of Lemma 4 from [31], due to the Lipschitz continuity of projection dynamics, rendering non-expansive property $\|P_M(x) - P_M(y)\| \le \|x-y\|, \forall x, y$, the above inequality (31) can be modified as

$$\begin{aligned}\dot{V} \le &-d\underline{\alpha}\Upsilon\gamma_1\|\tilde{\theta}\|^2 - (1-d)\varepsilon\|x-\bar{x}\|^2\\ &-(1-d)\varepsilon(x-x^\star)^{\mathrm{T}}\nabla J(x)\\ &+(1-d)\varepsilon\eta\|\nabla J(x)\|C\|\tilde{\theta}\| + (1-d)\varepsilon\eta\|x-x^\star\|C\|\tilde{\theta}\|\\ &+(1-d)\left(\|\nabla J(x)\| + \|(x-x^\star)\|\right)\|u_{\mathrm{ex}}(t)\|, \;\forall t \ge t_0+T. \end{aligned} \quad (32)$$

Provided Assumption 2 holds and utilizing the Cauchy-Schwarz inequality on the cross term, the above inequality (32) can be modified, $\forall t \ge t_0+T$

$$\begin{aligned}\dot{V} \le &-d\underline{\alpha}\Upsilon\gamma_1\|\tilde{\theta}\|^2 - (1-d)\varepsilon\Big[J(x) - J(x^\star) + \frac{\mu}{2}\|x-x^\star\|^2\Big]\\ &+(1-d)C^2\|\tilde{\theta}\|^2 + (1-d)\frac{\varepsilon^2\eta^2}{2}(L^2+1)\|x-x^\star\|^2\\ &+(1-d)(\beta_1+\beta_2)\|u_{\mathrm{ex}}(t)\|, \;\forall t \ge t_0+T \end{aligned} \quad (33)$$

where $\mu > 0$ and $\beta_1, \beta_2 \geq 0$ are the upper bounds on $\|\nabla J(x)\|, \|x - x^*\|$, respectively, $\forall x \in \bar{M}$. After some manipulation on (33), $\forall t \geq t_0 + T$, yields

$$\dot{V} \leq -\left(d\underline{\alpha}\Upsilon\gamma_1 - (1-d)C^2\right)\|\tilde{\theta}\|^2 - (1-d)\varepsilon\big(J(x) - J(x^\star)\big) - (1-d)\left(\varepsilon\frac{\mu}{2} - \frac{(L^2+1)}{2}\varepsilon^2\eta^2\right)\|x - x^\star\|^2 + \beta_3 e^{-\beta_4 t}. \quad (34)$$

Further, (34) can be modified as

$$\dot{V} \leq -\underbrace{\min\Big(2\big(\underline{\alpha}\Upsilon\gamma_1 - \frac{(1-d)}{d}C^2\big), \big(\varepsilon\mu - (L^2+1)(\varepsilon^2\eta^2)\big), \varepsilon\Big)}_{\beta_5 > 0} V + \beta_3 e^{-\beta_4 t} \quad (35)$$

where, the following necessary and sufficient conditions should be satisfied - $0 < \varepsilon < \frac{\mu}{\eta^2(L^2+1)}$, $\underline{\alpha}\Upsilon\gamma_1 > \frac{1-d}{d}C^2$; and β_3, β_4 are some computable positive scalars. Further using the comparison lemma (see [30], Lemma 3.4), $V(t)$ can be upper bounded as,

$$V(t) \leq \delta_1 e^{-\delta_2 t}, \ \forall t \geq t_0 + T \quad (36)$$

where $\delta_2 = \min(\beta_5, \beta_4)$ and δ_1 are computable positive scalars. Thus, $V(t)$ is exponentially converging to zero. ■

E. Boundedness of the Auxiliary Signals

Corollary 4.1. *Suppose that the results of Theorem 4 hold, then $Y(x(t)),\ Z(x(t)) \in \mathscr{L}_\infty$.*

Proof: From Theorem 4, it can be inferred that $x(t) \in \mathscr{L}_\infty$ which implies $\phi(x(t)) \in \mathscr{L}_\infty$ from Assumption 1. Utilizing the above facts, properties (8)-(10) of weighting function $\alpha(t)$ in (12), $\|Y(x(t))\|$ can be expressed as

$$\begin{aligned}\|Y(x(t))\| &= \|\int_{t_0}^{t} \underbrace{\alpha(r)}_{>0}\ \underbrace{\phi(x(r))\phi^{\mathrm{T}}(x(r))}_{\geq 0}\mathrm{d}r\| \\ &\leq \int_{t_0}^{t} \|\alpha(r)\|\|\phi(x(r))\phi^{\mathrm{T}}(x(r))\|\mathrm{d}r \\ &\leq \|\phi(x(t))\phi^{\mathrm{T}}(x(t))\|_{\max}\int_{t_0}^{t}\|\alpha(r)\|\mathrm{d}r \end{aligned} \quad (37)$$

which implies $Y(x(t)) \in \mathscr{L}_\infty$ since $\alpha(t) \in \mathscr{L}_1$ by definition. Based on algebraic relation $Z(x(t)) \triangleq Y(x(t))\theta$ in (11), it can be inferred that $Z(x(t)) \in \mathscr{L}_\infty$. ■

Remark 4. *In the proposed integral filter architecture (6-7) with the special properties of weighting function $\alpha(t)$, it can be observed that there is no unlearning of information since there is no forgetting factor involved unlike closed-loop filter architecture, requiring a computationally expensive switching mechanism for the update law through online IE verification [20], [22]. On the other hand, the introduction of weighting function $\alpha(t)$ also resolves the internal instability issue of open-loop filter architecture [20], [33], [34].* □

Remark 5. *Note that a more realistic situation regarding Assumption 2 can be considered i.e., if the performance index $J(x)$ is convex instead of strongly convex, while rest remaining same. Under this condition, it can be shown that objectives (4)-(5) are achieved asymptotically instead of exponentially.* □

IV. Simulation Results

The proposed AdESC algorithm is simulated using the following performance index or objective function $J(x) = (x_1 - 2)^2 + (x_2 - 2)^2$. Based on Assumption 1, the unknown constant parameter vector is $\theta = [1, 1, 0, -4, -4, 8] \in \mathbb{R}^6$ and the regressor matrix is $\phi(x(t)) = [x_1^2, x_2^2, x_1x_2, x_1, x_2, 1] \in \mathbb{R}^6$; The filter weighting function $\alpha(t) = exp^{-0.2(t-t_0)}$ is chosen such that it satisfies all the properties (8)-(10); the exploratory signal is selected as $u_{\mathrm{ex}}(t) = 0.25exp^{-0.2(t-t_0)}[1, sin(3t)]^{\mathrm{T}}$, which is a decaying signal, i.e., it helps to explore in the transient phase, while facilitates exploitation in the long term by diminishing itself. The control tuning parameter is chosen as $\varepsilon = 0.022, \eta = 1$. Figures 1-4 validate the claims of Theorem 4 i.e., trajectory $x(t)$ converges exponentially to the exact unknown minima $x^* = [2, 2]^{\mathrm{T}}$ and estimation error $\tilde{\theta}(t)$ exponentially converges to zero. In this scenario, the proposed IE-based AdESC algorithm outperforms the conventional PE-based AdESC algorithm $\dot{\hat{\theta}}(t) = -\gamma\Gamma_\theta\phi(x(t))\big(\hat{J}(x(t)) - J(x(t))\big)^{\mathrm{T}}$. In fact, it is evident from Figure 4 that the PE-based algorithm is moving in an opposite direction in the absence of PE condition (due to exponentially decaying exploratory signal), while the proposed IE-based algorithm is converging to the minima $(2, 2)$.

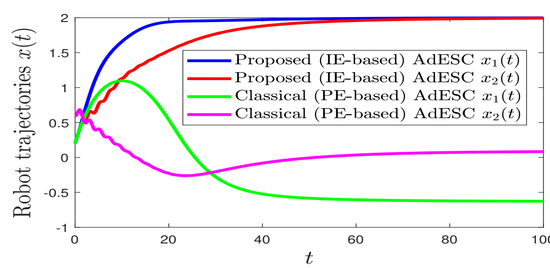

Fig. 1: The evolution of the state $x(t)$ and a comparison with different techniques.

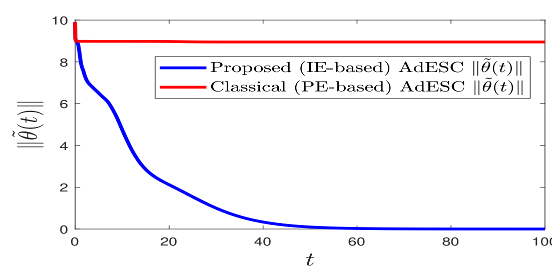

Fig. 2: The evolution of the norm of the parameter estimation error and a comparison with different techniques.

V. Conclusion and Future Work

This paper designs a novel PI-like AdESC algorithm for online optimization. Unlike past literature in adaptive ESC and data-driven adaptive ESC algorithms, where parameter convergence is ensured by restrictive conditions like the PE condition and rank condition on past stored data, the

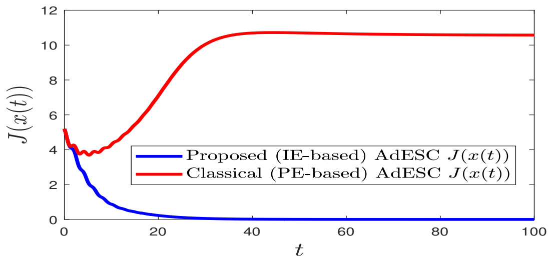

Fig. 3: The evolution of the performance index $J(x(t))$ and a comparison with different techniques.

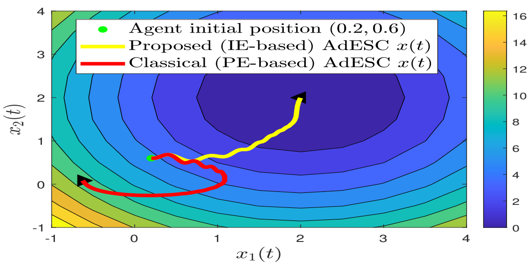

Fig. 4: The evolution of the trajectories in a phase plane plot to show the achievement of the optimization objective.

proposed AdESC algorithm ensures parameter convergence under a relaxed mathematical condition called Initial Excitation (IE). The novelty of the proposed PI-like AdESC algorithm relies on a newly introduced weighted integrator which omits the requirement of switching mechanism for rank-checking as well as internal instability issue unlike previous works on IE frameworks. Future work will focus on extending the formulation to multi-agent systems.

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